Third International Conference
Modelling and Development of Intelligent Systems
October 10 - 12, 2013
"Lucian Blaga" University
Sibiu - Romania

Knowledge about replenishable resources: the dynamics of unemployment and job creation

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Abstract

Defined in a general way resources are meant to enable the functioning of complex systems like human societies or biological and techno-economical networks. Apart from the primary inputs to such systems like raw materials, energy, raw information and finance there are a series of embedded or derived complex resources like housing, education and jobs which provide pivotal services. The latter may be often characterized by knowledge production and, on a more fundamental level, by time delays in otherwise quasi-continuously acting dynamical environment. We propose to analyze the role of time delays in order to better understand the dynamics of unemployment and job creation. Temporary and permanent jobs are highly context and delay-sensitive replenishable resources. Misunderstanding their dynamics can cause high and long lasting societal costs.

1 Introduction

In biology and in ecology many good examples of the fundamental role of time delays in otherwise continuous processes are found. A classical example is the Mackey-Glass delay equation which dates back to 1977 (see Lichtenberg and Lieberman [7] detailed explanations) and which describes the regeneration of (blood) cells. Being a scalar equation of type $\frac{dx(t)}{dt} = \frac{ax(t-\tau)}{1+[x(t-\tau)]^c} - bx(t)$, with $a, b, c > 0$ and $x([-\tau, 0])$ given, its dynamics is "harmless" for small τ but exhibits chaotic fluctuations for large enough $\tau > 0$ and thus offers an explanation for hard-to-control malfunctioning of biological regeneration processes. Knowledge about "too large delays" is also vital in the context of other replenishable complex resources, as shown by Nikolopoulos and Tzanetis [3] in a paper on the impact on loss of shelter due to catastrophic events like earthquakes. Other intriguing examples are discussed in the models of Misra and Singh ([1], [2]), where the replenishable resource refers to unemployment (job losses) and job creation. As a rule, different models may express some plausible valid facet of a complex empirical process like job loss and job creation. In the sequel we use computational approaches (symbolic, numerical) in order to support the process of domain knowledge extraction.

2 Some dynamic equations of labor and employment

The models of Misra and Singh ([1], [2]) describe the process of labor market fluctuations by a nonlinear dynamical model in four variables related to employment to be described in the sequel. It is based on the aforementioned model of Nikolopoulos and Tzanetis [3] which treats housing replenishment based on past information, the time delay of collecting reliable information being substantial. Note that when it comes to describe ways of measuring the labor markets, alternative classifications of variable or factors come to mind and they may pose some indeterminacies from the onset. This refers to e.g. what type of people to include into the class of the "unemployed", how to delimit different term structures of unemployment, etc.. In order to facilitate a model-based analysis of the employment process (in a developed economy) one may use a set of variables $x_i(t)$, $i = 1, ..., 4$, defined at each moment in time $t (t \geq 0)$ (this collection of variables may be further refined):

- $x_1(t) \geq 0$, the number of unemployed (including out of labor, part time ?),
- $x_2(t) \geq 0$, the number of persons with temporary employment,
- $x_3(t) \geq 0$, the number of persons with permanent employment, and
- $x_4(t)$, the vacancies or the newly created jobs (these may also be destroyed?)

During time evolution, We assume that a proportion of the unemployed may become permanently employed, and others temporarily employed. Furthermore some of the temporarily employed may become permanently employed. Finally, a part of both, the temporarily and permanently employed may lose their jobs and become unemployed. For simplicity, we also assume that all unemployed can initially cope with the tasks of any job on offer, but in time they are loosing skills owing to attrition. Also, we assume that there should be no barriers to job acceptance imposed by reduced individual mobility. Furthermore, as a crude approximation, a constant rate of growth of unemployment is assumed (owing for instance to the continuous action of labor saving technical progress). Migration rate of unemployed is assumed to be proportional to their number and the total number of vacant jobs which can be created are bounded and constant.

The time evolution of the umber of unemployed $x_1(t)$ is defined to depend on the following: (1) the rate of change of the number of the unemployed which will become permanently employed is proportional to $x_1(t)$ and to the number of permanent jobs $a_2 + x_4(t) - x_3(t)$, where $a_2 > 0$ is the total number of such permanent jobs available, and, (2) allowing for the transition from unemployment to temporary employment, their latter rate of change is proportional to $x_1(t)$ and to the number of temporary jobs available and vacant $a_4 + x_4(t) - x_2(t)$, where $a_4 > 0$ is the total number of temporary jobs available in the system. Hence, we have

$$\begin{aligned} \frac{dx_1(t)}{dt} &= a - a_1 x_1(t)(a_2 + x_4(t) - x_3(t)) - \\ &\quad - a_3 x_1(t)(a_4 + x_4(t) - x_2(t)) - a_5 x_1(t) + a_6 x_2(t) + a_7 x_3(t), \qquad (1) \\ &\quad \text{with initial condition } \; x_1(0) > 0. \end{aligned}$$

The coefficients $a_1, a_3 > 0$ are for scaling the single described effects and $a_5, a_6, a_7 > 0$ stand for migration rate of the unemployed, the transition rate of permanently and temporarily employed into the state of unemployment, respectively.

Turning now to the evolution of the temporarily employed persons, $x_3(t)$, we consider that the rate of change of unemployed into permanently employed will be proportional to $x_1(t)$ and to the number of vacant permanent jobs $a_4 + x_4(t) - x_2(t)$ (with $a_4 > 0$ being the number of vacant jobs permanently available). Again, the rate of transformation from being unemployed into temporarily employed is proportional to $x_1(t)$ and the number of temporary job vacancies $a_2 + x_4(t) - x_3(t)$, where $a_2 > 0$ is the number of total temporary job vacancies. Hence the differential equation for $x_2(t)$ reads:

$$\begin{aligned} \frac{dx_2(t)}{dt} &= a_3 x_1(t)(a_4 + x_4(t) - x_2(t)) - \\ &\quad - a_8 x_2(t)(a_2 + x_4(t) - x_3(t)) - a_9 x_2(t) - a_6 x_2(t), \\ &\quad \text{with initial condition } \; x_2(0) > 0. \end{aligned} \tag{2}$$

Coefficient $a_8 > 0$ is a constant of proportionality and the constant decay rate $a_9 > 0$ describes the exit of temporarily employed persons from the system (due to death, old age, or migration, see a similar argument for $a_{10} > 0$ below).

The rate of change of the number of unemployed which will find a permanent job is proportional to $a_1x_1(t)+a_8x_2(t)$ and the number of vacant job positions for permanent employment is $a_4+x_4(t)-x_3(t)$. Hence, the rate of change of the permanently employed $x_3(t)$ is given by the following differential equation:

$$\begin{aligned} \frac{dx_3(t)}{dt} &= (a_1 x_1(t) + a_8 x_2(t))(a_4 + x_4(t) - x_3(t)) + a_{10} x_3(t) - a_7 x_3(t), \\ &\quad \text{with initial condition } \; x_3(0) > 0. \end{aligned} \tag{3}$$

where the constant decay rate $a_{10} > 0$ describes the exit of permanently employed persons from the system (due to death, old age, or migration). Finally, the time evolution of newly created jobs is proportional to the time-delayed information about the unemployed in existence at $t - \tau$:

$$\begin{aligned} \frac{dx_4(t)}{dt} &= a_{12} x_1(t-\tau) - a_{13} x_4(t), \\ &\quad \text{with } \; x_1(\theta) = V(\theta), \;\; -\tau \le \theta \le 0, \quad \text{and with initial condition } \; x_4(0) > 0. \end{aligned} \tag{4}$$

Note that $V : R \to R$ is a differentiable function which has to be supplied by the modeler. The coefficients $a_{12}, a_{13} > 0$ are the rate of new job creation and the decay rate of permanent unemployment, respectively. The latter may be due to insufficient state funding or labor saving technical progress.

3 Equilibrium and bifurcation analysis

In order to analyze the dynamics of a higher dimensional system which escapes intuition some symbolic term reduction and manipulation are in order. The less consuming part (subsection 4) relates to determining the stationary points $0 = f_i(x_{10}, x_{20}, x_{30}, x_{40})$, $i = 1, ..., 4$, which allow for linearizing and simplifying around these points and upon which the qualitative dynamics of the system (stable orbits, oscillations) may be characterized. Determining the fate of the dynamics as a function of the time delay $\tau > 0$ which changes the eigenvalue spectrum of the linearized system is a more complicated symbolic procedure just touched upon in subsection 3.2 and which is based on Normal Form Theory of bifurcation analysis (for a recent account on theory and computational approaches consult Han and Yu [6]).

3.1 A unique equilibrium point in feasible region of the state space

Equilibrium or stationary points of the dynamic system 1–4 are solutions of the following system of algebraic (low order multinomial) equations:

$$\begin{array}{rcl} a - a_1x_1(a_2 + x_4 - x_3) - a_3x_1(a_4 + x_4 - x_2) - a_5x_1 + a_6x_2 + a_7x_3 & = & 0, \\ a_3x_1(a_4 + x_4 - x_2) - a_8x_2(a_2 + x_4 - x_3) - a_9x_2 - a_6x_2 & = & 0, \\ (a_1x_1 + a_8x_2)(a_4 + x_4 - x_3) + a_{10}x_3 - a_7x_3 & = & 0, \\ a_{12}x_1 - a_{13}x_4 & = & 0. \end{array} \tag{5}$$

Upon adding the equations of 5 we arrive at

$$a - a_5x_1 - a_9x_2 - a_{10}x_3 \ = \ 0. \tag{6}$$

From the 4th equation of 5 results

$$x_4 \ = \ \frac{a_{12}x_1}{a_{13}}, \tag{7}$$

and from equation 6 results

$$x_3 \ = \ \frac{a - a_5x_1 - a_9x_2}{a_{10}}. \tag{8}$$

By replacing x_3 and x_4 in the second and third equations of system 5 leads finally to a reduced system of two equations in two variables:

$$\begin{array}{rcl} f_3(x_1, x_2) & = & (a_1x_1 + a_8x_2)(a_2a_{13}a_{10} + a_{12}a_{10}x_1 - a_{13}(a - a_5x_1 - a_9x_2)) - \\ & & a_{13}(a_{10} + a_7)(a - a_5x_1 - a_9x_2) \ = \ 0, \\ f_4(x_1, x_2) & = & a_{10}a_3x_1(a_4a_{13} + a_{12}x_1 - a_{13}x_2) - \\ & & a_8x_2(a_2a_{13}a_{10} + a_{12}a_{10}x_1 - a_{13}(a - a_5x_1 - a_9x_2)) - \\ & & a_{10}a_{13}(a_6 + a_9)x_2 \ = \ 0. \end{array} \tag{9}$$

The two-dimensional system of equations 9 admits a positive solution x_{10}, x_{20}, which is depicted in figure 1 as the intersection of the graphs of $f_3(x_1, x_2) = 0$ and $f_4(x_1, x_2) = 0$. For a given set of parameter values

$a = 65000$	$a_1 = 0.00004$	$a_2 = 25000$	$a_3 = 0.0003$	$a_4 = 30000$	$a_5 = 2$	
$a_6 = 0.0009$	$a_7 = 0.0008$	$a_8 = 0.0001$	$a_9 = 0.9$	$a_{10} = 0.9$	$a_{12} = 0.9$	$a_{13} = 0.2$

we obtain the equilibrium point $(x_{10}, x_{20}) = (5520, 23370)$ and by using these values in 8 and 7 we obtain the third and forth coordinates of the equilibrium point, namely $(x_{30}, x_{40}) = (36585.55, 24840)$. It can be proven that this equilibrium point is unique for positive values of x_1 and x_2. However, this should be regarded as a simple case without claiming genericity for this to happen in most models of labor dynamics (indeed, it is not very difficult to state much simpler, empirically relevant nonlinear models with multiple equilibria, see e.g. Guckenheimer & Holmes [4]).

For obtaining information concerning the nature of the equilibrium point we compute the characteristic equation (eigenvalue equation) of the dynamical system 1–4 linearized at the equilibrium point. Using the (numerically) instantiated parameter values from above this finally results for the special case of $\tau = 0$ in the equation

$$((\lambda+11.97)(\lambda+2.98)(\lambda+3.45)-96.36-18.85\lambda)(\lambda+.2)-11.97+1.68(\lambda+2.98)(\lambda+3.46)+.5\lambda \ = \ 0$$

The solutions of this equation or the eigenvalues are (numerical values are rounded for convenience)

$$\lambda_1 \,;=; -13.37, \quad \lambda_2 \,;=; -4.41, \quad \lambda_3 \,;= \ -0.41 - 0.11i, \quad \lambda_4 \,;=; -0.41 + 0.11i$$

The eigenvalues have negative real parts which, in the context of our dynamical system, implies that the equilibrium point (x_{10}, x_{20}, x_{30}, x_{40}) is **asymptotically stable**, i.e. the equilibrium point is attracting any orbit starting in its vicinity.

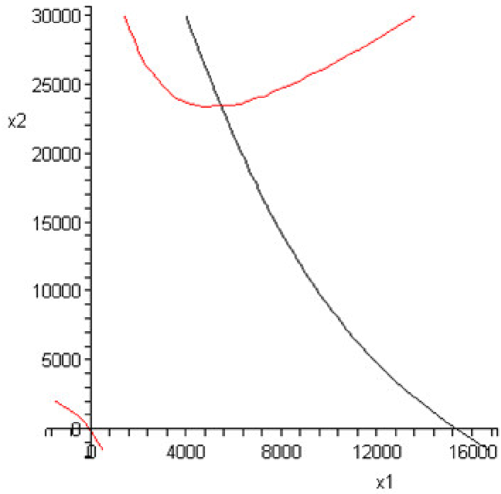

Figure 1: Graphs of the functions $f_3(x_1, x_2) = 0$ (red in colored display style) and $f_4(x_1, x_2) = 0$ (black in colored display style) intersect at a unique point which is the (x_1, x_2)-coordinate of the equilibrium.

3.2 Bifurcation analysis detects change in the dynamics

If, in contradistinction to subsection 4, we allow for delay $\tau > 0$ then the structure of the eigenvalues of a linearized system along the system trajectories may change. In order to capture this by means of detecting a qualitative change in the systems dynamics, a value τ_0 will be determined for which the system undergoes a Hopf bifurcation. This may be achieved by way of a symbolic computation which applies Normal Form Theory from bifurcation analysis (Han and Yu [6]). The procedure of determining such a $\tau_0(a, a_1, ..., a_{13}, .)$ is both potentially complex and tedious often resulting in long symbolic expressions. A Maple program developed in [6] for dealing symbolically with bifurcation analysis may be applied. In doing so a series of intermediate expressions will be generated. With b_{ij} standing for $\dfrac{\partial f_i(x)}{\partial x_j}$ computed at $x = (x_{10}, x_{20}, x_{30}, x_{40})$ we obtain the Jacobian of the dynamics at the stationary point. Formally we also differentiate with regard to $x_1(t - \tau)$, resulting in c_{41}, which in our simple linear case happens to be a_{12}. Hence upon executing in Maple commands

```
> b11:=eval(diff(F1,x1),[x1=x10,x2=x20,x3=x30,x4=x40]);
> ...
> b44:=eval(diff(F4,x4),[x1=x10,x2=x20,x3=x30,x4=x40]);
> c41:=eval(diff(F4,y1),[x1=x10,x2=x20,x3=x30,x4=x40]);
```

we get the Jacobian matrix and c_{41}, where "y1" stands for extra variable $x_1(t - \tau)$, $\tau > 0$. In order to get a Hopf bifurcation point in terms of $\tau > 0$ one executes the further Maple statements:

```
> alpha3:=-(b44+b11+b22+b44);
> alpha2:= b11*(b22+b33)+b22*b33-b12*b13-b12*b21-\
              -b32*b23+b44*(b11+b22+b33);
> alpha1:=-b44*(b11*(b22+b33)+b22*b33-b31*b13-b12*b21-\
              -b32*b23-b11*b22*b33+b12*b21*b33+b32*b23*b11+b31*b13*b22);
```

```
> alpha0:=-b44*(-b11*b22*b33+b12*b21*b33+b32*b23*b11+b31*b13*b22);

> beta2:=-b14;
> beta1:= b14*(b22+b33)-b13*b34-b24*b12;
> beta0:=-b12*b23*b34-b13*b24*b32-b14*b22*b33+b14*b23*b32+\
             +b13*b34*b22+b24*b12*b33;

> gamma6:=alpha3-2*alpha2;
> gamma4:=alpha2^2+2*alpha0-2*alpha1*alpha3-c41^2*beta2^2;
> gamma2:=alpha1^2-2*alpha2*alpha0-c41^2*beta1^2+2*beta0*beta2*c41^2;
> gamma0:=alpha0^2-c41^2*beta0^2;

> eq1:=x^8+gamma6*x^6+gamma4*x^4+gamma2*x^2+gamma0;
> sols1:=solve(eq1,x);

>#Take a positive solution of sols1
> omega:=sols1[2];
```

The resulting value of τ_0 is symbolically expressed as a function of the original model parameters $a, a_1, ..., a_{10}$, and a_{12}, a_{13} as all the Jacobian matrix entries b_{ij} are functions of some of the latter. Translating into a more readable form we finally have:

$$\tau_0 \;=\; \arccos\left(\frac{(\beta_0-\beta_2\omega^2)(\alpha_2\omega^2-\omega^4-\alpha_0)+\beta_1\omega(\alpha_3\omega^3-\alpha_1\omega)}{c_{41}((\beta_0-\beta_2\omega^2)^2+\beta_1^2\omega^2)}\right) \;+\; \text{constant}\times\frac{\pi}{\omega},$$

which is a rather complicated function of the original model parameters. Note that although $c_{41} := a_{12}$ appears explicitly in this function it also affects γ_0, γ_2 and γ_4 in the above Maple expressions. In order to appreciate the potential complexity of equivalent symbolic manipulations applied to possible model extensions, bear in mind that the simplicity of equation 4 also much simplifies the Maple reduction process show here.

For the numerically instantiated parameter values of the model a delay value of $\tau_0 = 20.05768477$ results for locating a bifurcation point. If $0 \leq \tau < \tau_0$, the solution of our dynamical system is asymptotically stable. For $\tau = \tau_0$ the solution exhibits a limit cycle and for $\tau > \tau_0$ the solution becomes unstable. More numerically oriented bifurcation analyzes than the above procedure based on Normal Form Theory may also be performed by using automatic differentiation and trajectory pursuit (Gukenheimer & Meloon [5]). For more details on the computer assisted determination of the equilibrium and bifurcation points of out dynamical system, the reader may contact the authors.

4 Modes of simulation as different views on the process

Numerical simulation is performed for alternative sets of initial conditions by using the model parameter values from subsection . In all case we use the initial function $V(-\tau \leq \theta \leq 0) = x_1(0)$, i.e. we use the initial value at $t = 0$ of the retarded variable. The software used in the sequel is Matlab 12 and Scilab 5.4.1 respectively.

Figure 2 depicts a dynamical process which is asymptotically stable in all four variables, oscillating with diminishing amplitudes.

Figure 3 depicts a dynamical process which is asymptotically stable in all four variables, oscillating with slowly diminishing amplitudes.

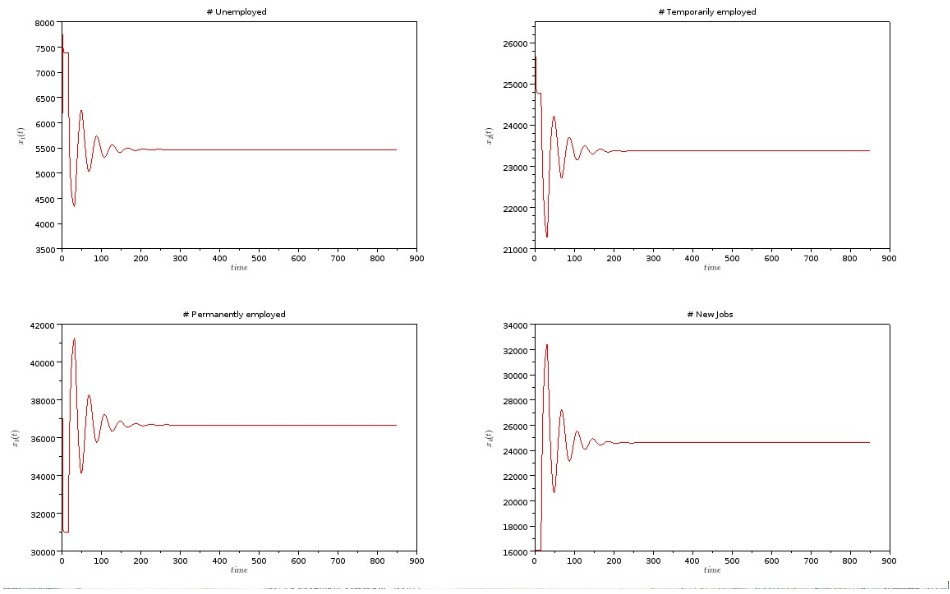

Figure 2: Time evolution of $x_i(t)$, $i = 1, ..., 4$ for the "small" delay $\tau = 15$.

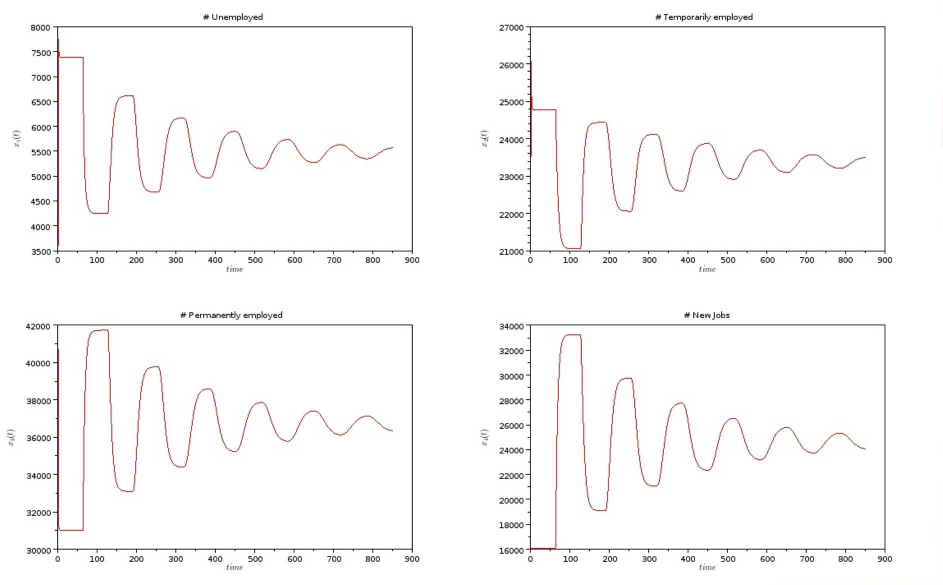

Figure 3: Time evolution of $x_i(t)$, $i = 1, ..., 4$ for the "large" delay $\tau = 63$.

We have also generated a stochastic version in the sense of Ito of the above differential equations by adding noise to the respective increments $\Delta_t x_i(t)$ independently to all $i = 1, ..., 4$ variables. Analyzing the similarity of these stochastic orbits (not shown here) to the empirical time series from labor markets of some developed economies is the aim of present research.

5 Conclusion

Our dynamical model is selected for representing certain aspects of the labor market in an advanced economy. In a preliminary phase, we consider the systemic effects of certain classes of stylized dynamic models which have been used in the literature for capturing a process which can be described as "replenishing a complex resource" in a biological or social context. This allows for viewing unemployment and job creation as being robustly described by a system of four nonlinear differential equations with time delay. As expected, the model shows that unemployment decreases if the number of newly created vacancies increases. Unemployment can be "controled" by by creating new jobs with a rate proportional to the number of unemployed (i.e. the process is not run-away; think of this as a feedback control).

A stochastic model version may be more useful for predictions. Such a model extension was tested and will be reported in future work. The existing model can be further generalized by taking into account some additional variables which describe job creation in the private and in the public sector. An interesting questions to pursue is what adaptations would be necessary if we wish to model a labor market from country with emerging economy, or those from trans-border and otherwise defined economic regions.

Acknowledgement: This paper is partially funded by the research Grant POSDRU/111/4.1/S/91816.

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